10 – Probability Distributions, Part II

A binomial experiment is a probability experiment with a number of repeated trials and the following properties:

- Each trial has two outcomes.
- The outcomes of each trial are independent of other trials.
- The probability of each specific outcome is uniform across trials.

Example 1: We roll a standard 6-sided die three times. Each time we roll the die, we record whether the die landed on a number less than 5, or not. That is, we call it a *success* when the die lands on a 1, 2, 3, or 4, and a *failure* when it lands on a 5 or 6. This is a binomial experiment.

The random variable in this experiment is the number of successes, and the possible values of the variable are 0, 1, 2, or 3.

The following notation will be used for binomial experiments:

Probability of success: p
Number of trials: n
Number of successes (variable): x

Example 1: In the example above, the probability of success is $p = \frac{4}{6} = \frac{2}{3}$, $n = 3$ is the number of trials (rolls of the die), and x could be 0, 1, 2, or 3.

The binomial distribution is the probability distribution for a binomial experiment. Making use of combinations to account for all possible scenarios, the following formula computes the probability of x successes in a binomial experiment: $p(x) = {}_nC_x\, p^x(1-p)^{n-x}$.

Example 1: If we wish to compute the probability that we roll 2 out of 3 successes, we use $p = \frac{2}{3}$, $n = 3$, and $x = 2$. That computation is $p(2) = {}_3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = (3)\left(\frac{4}{9}\right)\left(\frac{1}{3}\right) = \frac{4}{9} \approx 0.444$

x	$p(x)$
0	${}_3C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3 = (1)\,(1)\left(\frac{1}{27}\right) = \frac{1}{27} \approx 0.037$
1	${}_3C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = (3)\left(\frac{2}{3}\right)\left(\frac{1}{9}\right) = \frac{2}{9} \approx 0.222$
2	${}_3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^1 = (3)\left(\frac{4}{9}\right)\left(\frac{1}{3}\right) = \frac{4}{9} \approx 0.444$
3	${}_3C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = (1)\left(\frac{8}{27}\right)(1) = \frac{8}{27} \approx 0.270$

The complete probability distribution appears here. Note that the sum of the probabilities is equal to 1 (in this case it's not exactly 1 due to rounding).

The expected value of a binomial distribution is $E(x) = np$ and the standard deviation is $\sigma = \sqrt{np(1-p)}$.

Example 1: The expected number of successes in the example is $3\times\frac{2}{3} = 2$.

The standard deviation of the distribution is $\sqrt{3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)} = \sqrt{\frac{2}{3}} \approx 0.816$

Example 2: A recent poll concluded that 65% of college students identify with the Democratic Party. A sample of ten students is selected at random.

- Expected Value: We expect $10\times0.65 = 6.5$ students of then ten (on average) to identify with the Democratic Party.
- The probability that exactly half of the students (5 of the 10) in the sample identify with the Democratic Party is ${}_{10}C_5(0.65)^5(0.35)^5 \approx 0.154$, or a 15.4% chance.
- The probability distribution appears here.

x	$P(x)$
0	${}_{10}C_0(0.65)^0(0.35)^{10} \approx 0.00002$
1	${}_{10}C_1(0.65)^1(0.35)^9 \approx 0.0005$
2	${}_{10}C_2(0.65)^2(0.35)^8 \approx 0.004$
3	${}_{10}C_3(0.65)^3(0.35)^7 \approx 0.021$
4	${}_{10}C_4(0.65)^4(0.35)^6 \approx 0.069$
5	${}_{10}C_5(0.65)^5(0.35)^5 \approx 0.154$
6	${}_{10}C_6(0.65)^6(0.35)^4 \approx 0.238$
7	${}_{10}C_7(0.65)^7(0.35)^3 \approx 0.252$
8	${}_{10}C_8(0.65)^8(0.35)^2 \approx 0.176$
9	${}_{10}C_9(0.65)^9(0.35)^1 \approx 0.072$
10	${}_{10}C_{10}(0.65)^{10}(0.35)^0 \approx 0.013$