7 – Probability, Part II

Separate events within a probability experiment are said to be <u>independent events</u> if the outcome of one does not affect the probability of the other.

Example 1: If we flip two coins, the result of the first coin toss has no influence on the probability of either outcome of the second coin. So, two coin tosses are independent.

Example 2: If we pull a card from a deck, and the pull second, the probability that the second is an Ace depends on whether or not that first card was also an Ace. So, these are dependent events.

When two events, *A* and *B*, are independent, the probability of both happening is *p(A* and *B) = p(A)×p(B).*

Example 3: In a standard deck of playing cards, there are 52 cards in total with thirteen different face values: 2, 3, . . . 10, J, Q, K, A. Each of these cards has a *suit* that is either a heart (♥), a club (♣), a spade (♠), or a diamond (♦). Thus, there are thirteen cards of each suit.

The probability that a King is pulled from a deck of cards is $p(K) = \frac{4}{52} = \frac{1}{13}$. If that first card is replaced into the deck, then the probability of then pulling an Ace is $p(A) = \frac{4}{52} = \frac{1}{13}$. So, the probability that a King is pulled and then an Ace is pulled *with replacement* is $p(K \text{ and } A) = p(K) \times p(A) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169} \approx$ 0.0059

The conditional probability *p(B|A),* expressed as the "probability of *B*, given *A*" represents the probability of *B* happening once *A* is known to either have already happened, or to be guaranteed to happen. Note: If A and B are independent, then *p(B|A) = p(B).*

Example 4: If we do not replace the first card, the probability of the second card is altered. For example, Once a King has been selected, there are 51 cards remaining (3 of which are Kings), and so the probability that a second King is pulled from a deck, after the first King was pulled, is $p(K|K) = \frac{3}{51}$.
Similarly, $p(K|\text{non} - K) = \frac{4}{51}$

When two events, *A* and *B*, are not independent, then, *p(A* and *B) = p(A)×p(B|A).*

Example 5: From a standard deck of playing cards, the probability that a King is dealt followed by an Ace is *p(K and A) = p(K)×p(A|K)* = $\frac{1}{13} \times \frac{4}{51} = \frac{4}{663} \approx 0.0060$.

In order to continue into more complicated probabilities, we must first introduce some counting techniques.

The *Fundamental Principle of Counting (FPC)* is used when we wish to count the number of ways a sequence of tasks can be performed. Dealing 5 cards in poker is an example of such a sequence of tasks, though it may be easiest to understand the FPC with a simpler example:

Example 6: In a deli, a sandwich is custom ordered. There are 5 varieties of bread, 5 options for meat, and 4 options for cheese, as shown below:

Bread	**Meat**	**Cheese**
White	None	Provolone
Wheat	Turkey	Swiss
Rye	Ham	American
Italian	Roast Beef	Cheddar
Pumpernickel	Corn Beef	

> If we begin to count sandwiches, we would start with one with white bread and no meat. There would be four sandwiches like this, because each cheese option would make for a different sandwich. There would similarly be four sandwiches with white bread and turkey, four with white bread and ham, and so on. In total, the number of sandwiches with white bread would be 20, which comes from multiplying number of meat choices and the number of cheese choices. Further, there would be 20 sandwiches with wheat bread, 20 with rye, and so on. In total, there would be 100 different choices, which comes from the product 5×5×4.

The statement of the *FPC* is confusing because it is a generalization of the idea in the example above.

In words, it is this: The number of ways to perform a sequence of tasks is equal to the product of the numbers of ways to perform each individual task.

Or, in symbols, it is this: If tasks $t_1, t_2, t_3, \ldots t_n$ can each be performed in $k_1, k_2, k_3, \ldots k_n$ ways, then the sequence of all n tasks done in succession can be done in $k_1 \times k_2 \times k_3 \times \ldots \times k_n$ ways.

Sometimes the tasks in counting problems are related to each other. For example, if students were being selected from a class, then after the first student is selected there are fewer students available to choose from for the second student.

> Example 7: Five students are chosen from a class of 35 students. The first one chosen will receive an A on the next exam. The second, a B, and so on. The fifth student selected will receive an F. If we wish to count the different ways to select five students in this way, we must use the FPC. There are 35 options for the first student, who receives an A. Then, there are 34 remaining for the B. In this way, we will have 35×34×33×32×31 = 38,995,840 distinctly different ways to choose five students.

If the order in which the students are chose is irrelevant, then the problem is different.

> Example 8: Five students will be chosen from a class of 35 students. They will all receive As on the next exam. We already know that there are 35×34×33×32×31 = 38,955,840 different ways to choose five students. However, the order of selection will produce many repeated groups of five students.
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> For instance, consider these 5 specific students in the class mentioned above: Arnold, Betty, Caleb, Donnica, and Edward. In one on those nearly 39 million groups, these five students are in the order above (A-B-C-D-E). In another, the same five students are ordered (B-A-C-D-E). In total, these five students will be grouped together 5×4×3×2×1 = 120 times. So, we must divide to get an accurate number: $\frac{38955840}{120}$ = 324,632 combinations. This is how many different groups of five students there are when the order of selection is not important.

In these scenarios we have two short-cuts to the FPC: *permutations* and *combinations*. Permutations are specifically ordered arrangements of objects. Combinations are unordered groups of objects. One way to remember the difference is that O and P (as in, *order* and *permutations*) are next to each other in the alphabet.

For any positive integer n, the product $n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ is known as n-factorial, denoted $n!$. Factorials are useful in writing the formulas for permutations and combinations.

The number of ways to order r objects chosen from a group of n objects is ${}_nP_r = \frac{n!}{(n-r)!}$. These are permutations.

The number of ways to chose r objects, in any order, chosen from a group of n objects is ${}_nC_r = \frac{n!}{(n-r)!r!}$. These are combinations.

Example 8: Consider the letters A, B, C, D, and E. The number of arrangements of any three of those letters is called permutations. There are ${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1}$ = 5×4×3 = 60.

Those 60 permutations are listed below.

ABC	ABD	ABE	BCD	BCE	CDE	ACD	ACE	ADE	BDE
ACB	ADB	AEB	BDC	BEC	CED	ADC	AEC	AED	BED
BAC	BAD	BAE	CBD	CBE	DCE	CAD	CAE	DAE	DBE
BCA	BDA	BEA	CDB	CEB	DEC	CDA	CEA	DEA	DEB
CAB	DAB	EAB	DBC	EBC	ECD	DCA	EAC	EAD	EBD
CBA	DBA	EBA	DCB	ECB	EDC	DAC	ECA	EDA	EDB

Example 9: Using the same 5 letters, the number of combinations of three letters is equal to

${}_5C_3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = \frac{5\times4\times3\times2\times1}{2\times1\times3\times2\times1} = 10$. Those 10 combinations (of arbitrary order) are listed below.

ABC	ABD	ABE	BCD	BCE	CDE	ACD	ACE	ADE	BDE