

## 9 – Probability Distributions, Part I

A random variable is a variable with outcomes determined by chance. The information in this section will emphasize quantitative random variables.

Example 1: If you flip three coins, the number of heads is a random variable with outcomes  $\{0, 1, 2, 3\}$ .

Example 2: If you roll two dice, the result is a random variable with outcomes  $\{2, 3, 4, \dots, 12\}$ .

A probability distribution for a random variable is a table that lists all possible outcomes of the variable along with each outcome's associated probability.

Examples 1 & 2: The probability distributions for the number of heads when flipping three coins ( $x$ ) and the result of rolling two dice ( $y$ ) are shown below.

$x$	$p(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

$y$	$p(y)$
2	$\frac{1}{36}$
3	$\frac{1}{18}$
4	$\frac{1}{12}$
5	$\frac{1}{9}$
6	$\frac{5}{36}$
7	$\frac{1}{6}$
8	$\frac{5}{36}$
9	$\frac{1}{9}$
10	$\frac{1}{12}$
11	$\frac{1}{18}$
12	$\frac{1}{36}$

Because of the way probability is computed, the following properties apply to all probability distributions:

- The sum of all the probabilities is equal to 1.
- Each probability is between 0 and 1

The expected value (aka, mean; notation:  $E(x)$ ) of a probability distribution is a theoretical value that is computed by multiplying each outcome by its associated probability, and then adding these products. The expected value represents the value of the random variable one should expect, or the mean of the random variable's outcomes when repeated many times.

Example 1: The expected value of the random variable  $x$  above is  $E(x) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 1.5$ . Of course, there is no way to get 1.5 heads when three coins are flipped. But if three coins were flipped many times we expect an average of 1.5 for the number of heads for all of the trials.

The standard deviation of a probability distribution is a theoretical value that is computed by multiplying the square of each outcome by its associated probability, adding these products, subtracting the mean, and then taking the square root of the result. You will not be asked to compute the standard deviation in this class.

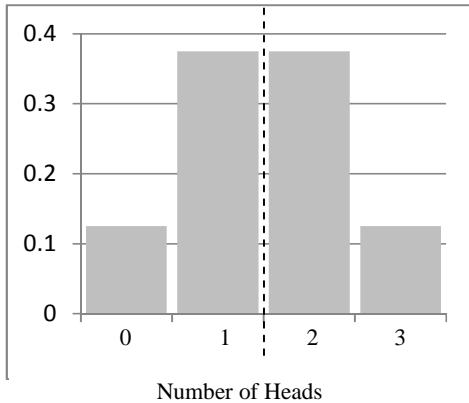
Example 1: The standard deviation of the random variable  $x$  above is  $\sqrt{0^2\left(\frac{1}{8}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{3}{8}\right) + 3^2\left(\frac{1}{8}\right) - (1.5)^2} = 0.866$ . This gives us an idea of the spread of the outcomes for this random variable. For example, most

trials of this random variable will have results within one standard deviation of the mean, or between 0.634 and 2.366.

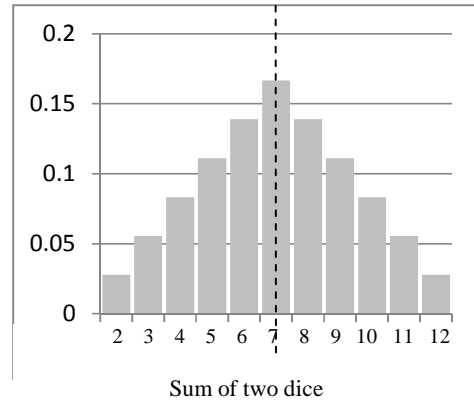
Visual displays of probability distributions are typically bar graphs.

Examples 1 & 2: The probability distributions above can be represented by bar graphs shown below.

**Probability Distribution for Number of Heads**



**Probability Distribution for Dice Outcome**



When probability distributions are symmetric, the expected value will occur at the center of the distribution. In the images above, a dashed line shows the center. The first image confirms the above computation of 1.5, and the second image shows that the expected value when rolling two dice is 7.

Expected value is a useful computation when analyzing games of chance. The example below shows how.

Example 3: A local charity is selling raffle tickets. In total, 1000 tickets were sold for \$1 each, and one ticket holder wins \$350. Find the expected value of the raffle.

*Solution Method 1:* We can set up a probability distribution for the amount gained by ticket holders ( $x$ ). The expected value of the game is then

$$E(x) = -1\left(\frac{999}{1000}\right) + 349\left(\frac{1}{1000}\right) = -0.65$$

*Solution Method 2:* Fundamentally, the expected value is the mean amount gained or lost on the raffle. Since \$999 was lost on all of the losing tickets and \$349 was gained by the single winner, from the perspective of all 1000 ticket holders that is a total loss of \$650, or \$0.65 each.

The expected value is -\$0.65

Event	$x$ (gain)	$p(x)$
Lose	-1	$\frac{999}{1000}$
Win	349	$\frac{1}{1000}$

A fair game is one with an expected value of zero. No casino games or lotteries are fair games. Otherwise they would not turn a profit and would be unsustainable.

Example 3: Instead of charging \$1 for tickets in the raffle above, what should the price of a ticket be in order to make the raffle a fair game?

*Solution:* Let the price of a ticket be \$ $C$ , for some number  $C$ . Then, the gain of a winning ticket is  $350 - C$ , and the gain of a losing ticket is  $-C$ . The probability distribution is shown here, and the equation for expected value will enable us to find  $C$ .

$E(x) = 0 = -C\left(\frac{999}{1000}\right) + (350 - C)\left(\frac{1}{1000}\right)$ . The solution to this equation is  $C = 0.35$ . So, if 35 cents is charged for each ticket, then the raffle is a fair game.

Event	$x$ (gain)	$p(x)$
Lose	$-C$	$\frac{999}{1000}$
Win	$350 - C$	$\frac{1}{1000}$